

THE DECATEGORIFICATION OF BORDERED KHOVANOV HOMOLOGY

LAWRENCE P. ROBERTS

ABSTRACT. In [14], [15] the author showed how to decompose the Khovanov homology of a link \mathcal{L} into the algebraic pairing of a type D structure and a type A structure (as defined in bordered Floer homology), whenever a diagram for \mathcal{L} is decomposed into the union of two tangles. Since Khovanov homology is the categorification of a version of the Jones polynomial, it is natural to ask what the type A and type D structures categorify, and how their pairing is encoded in the decategorifications. In this paper, the author constructs the decategorifications of these two structures, in a manner similar to Ina Petkova's decategorification of bordered Floer homology, [13], and shows how they recover the Jones polynomial. We also give a new proof of the mutation invariance of the Jones polynomial which uses these decomposition techniques.

1. BACKGROUND AND MOTIVATION

In [7], M. Khovanov describes a homology theory whose “Euler characteristic” is a reparametrization of the Jones polynomial of an oriented link \mathcal{L} in S^3 . In particular, he uses a link diagram L for \mathcal{L} and defines a bigraded, free Abelian group $\langle\langle L; \mathbb{Z} \rangle\rangle^{*,*}$ whose generators are decorated resolutions of L . The group $\langle\langle L; \mathbb{Z} \rangle\rangle^{*,*}$ admits a $(1, 0)$ differential ∂_{KH} for which the homology $\text{KH}^{*,*}(L, \mathbb{Z})$ is an invariant of the link \mathcal{L} .

Since the differential is $(1, 0)$ we can decompose $\langle\langle L; \mathbb{Z} \rangle\rangle^{*,*}$ as a direct sum of chain complexes $\bigoplus_{j \in \mathbb{Z}} (C^{*,j}, \partial_{\text{KH}})$ where $C^{*,j}$ is the subgroup of $\langle\langle L; \mathbb{Z} \rangle\rangle^{*,*}$ whose second grading, called the quantum grading, is equal to j . The “Euler characteristic” of $\langle\langle L; \mathbb{Z} \rangle\rangle^{*,*}$ is taken to be the polynomial

$$J_L(q) = \sum_{j \in \mathbb{Z}} \chi(C^{*,j}) q^j$$

where $\chi(C^{*,j})$ is the usual Euler characteristic of a finitely generated chain complex with free chain groups. This polynomial has the form $(-1)^{n_-(L)} q^{(n_+ - 2n_-)(L)} \tilde{J}_L(q)$, where $n_{\pm}(L)$ is the number of positive/negative crossings in L and $\tilde{J}_L(q)$ can be computed from the unoriented link diagram. If we let U be an unknot, and $L \sqcup U$ be

a link with an unlinked and unknotted component U , then $\tilde{J}_L(q)$ satisfies

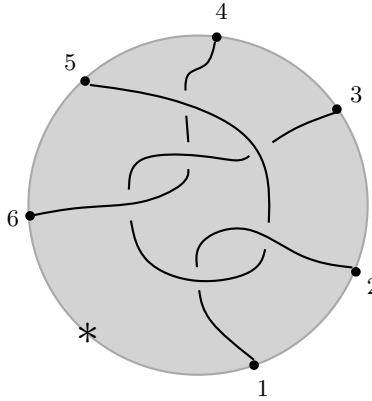
$$\begin{aligned}\tilde{J}_U(q) &= q + q^{-1} \\ \tilde{J}_{L \sqcup U}(q) &= (q + q^{-1})\tilde{J}_L \\ \tilde{J}_L &= \tilde{J}_{L_0} - q\tilde{J}_{L_1}\end{aligned}$$

Thus $J_L(q)$ is a version of the unnormalized Jones polynomial.

Due to this relationship, made more precise below, Khovanov homology is called a *categorification* of the Jones polynomial, and the polynomial is called the *decategorification* of Khovanov homology. These terms are used since the polynomial arises from a process more general than just taking Euler characteristics. Despite both the homology and its decategorification giving link invariants, the homology $\langle\langle L \rangle\rangle$ is known to be stronger than the Jones polynomial, and has many additional properties related to the cobordism on links.

In [14], [15] the author, inspired by bordered Floer homology, describes a similar construction for tangles¹. The formal algebraic structure of these tangle invariants mimics the formal structure of Ozsváth, Lipshitz, and Thurston's description of bordered Floer homology, [11], which also provided a road map for constructing a gluing theory for Khovanov homology.

The construction in [15] takes a tangle diagram T in a disc D , with a marked point $*$ $\in \partial D$:

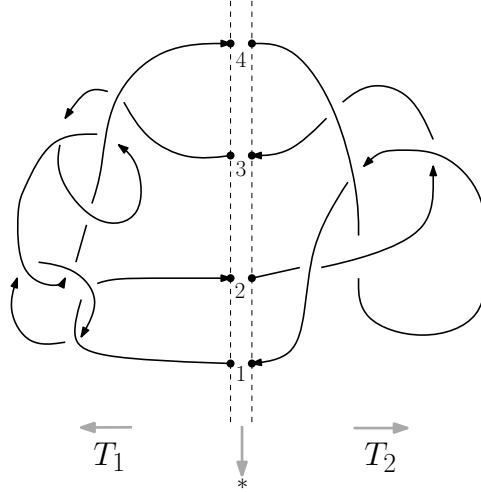


and associates to it a bigraded differential module $\langle\langle T \rangle\rangle$ over a differential bigraded algebra $\mathcal{B}\Gamma_n$, where T has $2n$ endpoints on ∂D . Each differential is $(1, 0)$ in the respective bigradings, and the action $\langle\langle T \rangle\rangle \otimes_I \mathcal{B}\Gamma_n \rightarrow \langle\langle T \rangle\rangle$ preserves the bigradings on its source and target. The homotopy type of $\langle\langle T \rangle\rangle$ as an A_∞ -module is invariant under the three Reidemeister moves applied to swatches in the diagram T , and thus

¹There are several other constructions of Khovanov homology for tangles, [8], [1], [2], [4], [10]

defines a tangle invariant.

The main goal in defining these algebraic structures is to obtain a complete gluing theory for Khovanov homology. For instance, let L be a link diagram in an oriented sphere S , and $S = D_1 \cup D_2$, where each closed disk D_i inherits its orientation from S and $D_1 \cap D_2 = C$ is a circle in S . We will orient C as the boundary of D_1 . Suppose L is transverse to C away from its crossings. Then $L \cap D_1 = T_1$ and $L \cap D_2 = T_2$ are tangles in oriented disks. If we choose $*$ $\in C$, then each disk also has a marked point in its boundary. We illustrate this situation in the diagram below, where we have taken $*$ to be the point at infinity. D_1 is then the left half plane, and D_2 is the right half plane.



To (D_1, T_1) assign the bigraded differential module $\langle\langle T_1 \rangle\rangle$ described above. To (D_2, T_2) the construction in [14] assigns a bigraded Abelian group $\llbracket T_2 \rrbracket$ and a $(1, 0)$ map $\delta : \llbracket T_2 \rrbracket \longrightarrow \mathcal{B}\Gamma_n \otimes_I \llbracket T \rrbracket$, which satisfies the structure relation for a type D structure, [11].

There is an algebraic construction \boxtimes which pairs differential graded modules and type D structures to obtain a normal chain complex. This can be adapted to the bigraded setting and applied to compute $\langle\langle T_1 \rangle\rangle \boxtimes \llbracket T_2 \rrbracket$. The main result of [15] is that $\langle\langle T_1 \rangle\rangle \boxtimes \llbracket T_2 \rrbracket$ is chain homotopy equivalent to $\langle\langle L \rangle\rangle$, the original link diagram. Furthermore, \boxtimes preserves chain homotopy equivalence when we alter either factor by a homotopy equivalence (in the respective categories of A_∞ -modules and type D structures), so this effects a gluing of the *homotopy types* of the invariants assigned to the tangles. As it is the homotopy types that are invariants of the tangles and links, and not the raw algebraic object, this provides a complete gluing theory for Khovanov homology.

Having defined $\langle\langle T_1 \rangle\rangle$ and $\llbracket T_2 \rrbracket$ it is natural to ask if there is any analog of these

objects, and their pairing, in the simpler world of the Jones polynomial. In this paper we provide the answer to this questions by describing the decategorifications of $\llbracket T \rrbracket$ and $\llbracket T \rrbracket$. The next section elaborates on the notion of decategorification we will use. In section 3 the decategorification is described in the abstract. The section 4 we provide more detail about the algebras involved, which allows us to give a concrete description of the decategorification in section 5. This will all be described for the type A structure. In section 6 we show how to interpret these results for type D structures. Then we can show how to recover the Jones polynomial in section 7 and reprove its invariance under mutation in section 8.

The concrete description can be used to generalize this paper to other settings, and exhibit a kind of planar algebra structure, a topic the author pursues in a sequel, [16].

Convention: If M is a bigraded module, then $M\{(m, n)\}$ is the bigraded module with $(M\{(m, n)\})_{i,j} = M_{i-m, j-n}$.

2. BACKGROUND ON DECATEGORIFICATION

First, we provide a little more detail concerning the method. Above, we referred to the polynomial $J_L(q)$ as a “Euler characteristic.” A better, and more sophisticated version of this statement can be obtained through the use of Grothendieck groups. For this paper, we will need a generalization of the following version of the Grothendieck construction, taken from [9]:

Definition 1. *Let A be a bigraded associative algebra. The Grothendieck group $K_0(A)$ is the $\mathbb{Z}[q, q^{-1}]$ -module generated by the elements $[P]$ where P is a finitely generated, bigraded, projective A -module, and subject to the relations that $[P] = [P'] + [P'']$ when there is a short exact sequence $0 \rightarrow P' \rightarrow P \rightarrow P'' \rightarrow 0$, and $[P\{(m, n)\}] = (-1)^m q^n [P]$.*

When A is \mathbb{Z} (in bigrading $(0, 0)$) the relations above confirm that $[\llbracket L; \mathbb{Z} \rrbracket^{*,*}] = \sum_j [C^{*,j}] = \sum_j q^j [C^{*,j}\{(0, -j)\}]$. Now $C^{*,j}\{(0, -j)\}$ is in gradings $(m, 0)$ for $m \in \mathbb{Z}$. Thus, $[C^{*,j}\{(0, -j)\}] = \sum_i (-1)^i [C^{i,j}\{(-i, -j)\}]$. Since $\llbracket L; \mathbb{Z} \rrbracket^{*,*}$ is free, so is $C^{i,j}$ so the image of $C^{i,j}\{(-i, -j)\}$ (in bigrading $(0, 0)$) in the Grothendieck group is just the rank of $C^{i,j}$. Thus,

$$[\llbracket L; \mathbb{Z} \rrbracket^{*,*}] = \sum_{i,j \in \mathbb{Z}} (-1)^i q^j \text{rk } C^{i,j}$$

from which it readily follows that $[\llbracket L; \mathbb{Z} \rrbracket^{*,*}] = J_L(q)$.

This can be extended to the case where P is a finitely generated, bigraded projective A -module with a $(1, 0)$ -differential. Following the pattern, the decategorifications of $\llbracket T \rrbracket$ and $\llbracket T \rrbracket$ should be the elements in a Grothendieck group for the differential

bigraded algebra $\mathcal{B}\Gamma_n$ where n is the number of unclosed components in T .

Let A be a bigraded differential graded algebra with $(1, 0)$ differential. We will consider *right* differential graded modules over A , since that is the structure of $\langle\langle T \rangle\rangle$ over $\mathcal{B}\Gamma_n$. Following [9], we aim to define the Grothendieck group $K_0(A)$. We need some additional definitions.

Definition 2. $\mathcal{K}(A)$ is the triangulated category found from the category of (right) bigraded differential modules with $(1, 0)$ differential by quotienting out by the morphisms homotopic to zero. $\mathcal{D}(A)$ is the derived category found by localizing $\mathcal{K}(A)$ at its quasi-isomorphisms.

Here it is understood that the homotopies, chain maps, etc. do not change the quantum (second) grading. The distinguished triangles in the triangulated structure are those diagrams triangle isomorphic to a standard triangle

$$M \xrightarrow{u} N \longrightarrow C(u) \longrightarrow M\{(-1, 0)\}$$

where u is a morphism of differential graded modules, and $C(u)$ is the mapping cone of u : the module $N \oplus M\{(-1, 0)\}$ equipped with the differential

$$\begin{bmatrix} d_N & u \\ 0 & -d_M \end{bmatrix}$$

Definition 3. A differential graded module P is said to be projective if, given any (right) differential graded module M with trivial homology, the complex $\text{Hom}(P, M)$, defined in [5], has trivial homology.

Definition 4. $\mathcal{KP}(A)$ is the full subcategory of $\mathcal{K}(A)$ whose objects are the projective differential graded modules.

In Part II, section 10 of [5] a bar construction is described which takes any differential graded module M and finds a quasi-isomorphic projective differential graded module $B(M)$. Thus, $\mathcal{KP}(A)$ is equivalent to the derived category $\mathcal{D}(A)$.

Definition 5. An object M of $\mathcal{KP}(A)$ is compact if the natural inclusion

$$\bigoplus_{i \in I} \text{Hom}(M, N_i) \hookrightarrow \text{Hom}(M, \bigoplus_{i \in I} N_i)$$

is an isomorphism for any collection of objects $\{N_i \mid i \in I\}$.

Definition 6. $\mathcal{P}(A)$ is the full subcategory of $\mathcal{K}(A)$ whose objects are compact, projective modules over A .

We are now in a position to define $K_0(A)$.

Definition 7. Let A be a bigraded differential graded algebra with $(1, 0)$ differential. $K_0(A)$ is the Grothendieck group of the category $\mathcal{P}(A)$. More specifically, $K_0(A)$ is the Abelian group with a generator $[P]$ for each compact, projective differential bigraded (right) module P over A , with $(1, 0)$ differential, subject to the relations $[P_2] = [P_1] + [P_3]$ for each distinguished triangle $P_1 \rightarrow P_2 \rightarrow P_3$

It follows from the definition that $[P\{(1, 0)\}] = -[P]$ (from the distinguished triangle coming from the mapping cone of the identity on P), and that $K_0(A)$ is a $\mathbb{Z}[q, q^{-1}]$ -module, where the action is $q^i[P] = [P\{(0, i)\}]$. In particular, $[P\{(m, n)\}] = (-1)^m q^n [P]$.

3. THE GROTHENDIECK GROUP OF $\mathcal{B}\Gamma_n$

We apply this construction to $\mathcal{B}\Gamma_n$. More details about this algebra will be given below. For this computation all that is required are the following properties, [14]:

- (1) $\mathcal{B}\Gamma_n$ is the quotient of a quiver algebra $\mathcal{Q}\Gamma_n$ defined by an *acyclic* directed graph Γ_n ,
- (2) The relations defining this quotient consist of identities involving paths of length ≥ 1 ,
- (3) The $(1, 0)$ differential is non-trivial only on paths of length ≥ 1 .

Let I_{2n} be the algebra of (orthogonal) idempotents in $\mathcal{B}\Gamma_n$. We will now prove that

Theorem 8. $K_0(\mathcal{B}\Gamma_n)$ is isomorphic to the $\mathbb{Z}[q^{1/2}, q^{-1/2}]$ module spanned by the idempotents corresponding to vertices in Γ_n .

Note that this is in keeping with the computation of Grothendieck groups for acyclic quiver algebras with relations, [6]. We have switched to $\mathbb{Z}[q^{1/2}, q^{-1/2}]$ since the quantum grading on $\mathcal{B}\Gamma_n$ is half-integral, but no other changes are necessary to the above construction.

Proof: Let P be a right bigraded differential module over $\mathcal{B}\Gamma_n$. Since P is projective and compact in $\mathcal{K}(\mathcal{B}\Gamma_n)$ we may use a homotopy equivalent representative of P which is finitely generated. Since Γ_n is acyclic and directed, there is a vertex v which has no out edges. Let I_v be the corresponding idempotent. now consider PI_v . This is a submodule of P since the action of any element of $\mathcal{B}\Gamma_n$ arises as the image of the action of path elements in $\mathcal{Q}\Gamma_n$. As no path starts at v , the only element which acts non-trivially on PI_v is I_v . Furthermore, PI_v is a subcomplex of P with its $(1, 0)$ differential since

$$d_P(p \cdot I_v) = d_P(p) \cdot I_v + (-1)^* p \cdot d_{\mathcal{B}\Gamma_n} I_v = d_P(p) \cdot I_v$$

for any $p \in P$. Thus the image of any element in PI_v under d_P will be in PI_v . Thus there is a distinguished triangle

$$P/PI_v\{(1, 0)\} \longrightarrow PI_v \longrightarrow P \longrightarrow P/PI_v$$

in $\mathcal{K}(\mathcal{B}\Gamma_n)$. Therefore, $[PI_v] = [P] + [P/PI_v\{(1,0)\}] = [P] - [P/PI_v]$, so $[P] = [PI_v] + [P/PI_v]$. Now P/PI_v is still a module over the differential graded algebra $\mathcal{B}\Gamma_n$, but only the elements of $\mathcal{Q}\Gamma'$ will act non-trivially, where Γ' is the directed graph $\Gamma_n \setminus \{v\}$. Since Γ' is acyclic, it also has a vertex v' with no outward edges. Due to the orthogonality of the idempotents $(P/PI_v) \cdot I_{v'} = P/PI_{v'}$. Applying this reasoning repeatedly, and using that P is finitely generated, we arrive at

$$[P] = \sum_{v \in \text{vert}(\Gamma_n)} [PI_v]$$

Since the action of $\mathcal{B}\Gamma_n$ is essentially trivial on PI_v , we can consider PI_v to be a bigraded complex over \mathbb{Z} with $(1,0)$ differential, just as above. Thus $[PI_v] \in \mathbb{Z}[q^{1/2}, q^{-1/2}][P_v]$ where P_v is the module with a \mathbb{Z} in bigrading $(0,0)$, trivial differential, and $P_v I_v = P_v$. Thus we see that the $\mathbb{Z}[q^{1/2}, q^{-1/2}]$ module spanned by the idempotents of $\mathcal{B}\Gamma_n$ maps surjectively to $K_0(\mathcal{B}\Gamma_n)$.

Before continuing we note that this is precisely what happens for bound quiver algebras, [6], as they too have Jordan-Hölder sequences of this type.

We show that this map is an isomorphism by constructing a module P for which $[P] = \sum_{v \in \text{vert}(\Gamma_n)} J_v(q)[P_v]$ for any collection of polynomials $J_v(q)$ with integer coefficients. For each term $\pm a q^j$ we have a copy of $P_v^a\{(0,j)\}$ if the sign is $+$, and $P_v^a\{(1,j)\}$ if the sign is $-$. We take the direct sum over all such terms. The differential is taken to be trivial. We do this for all $v \in \text{vert}(\Gamma_n)$, and then define the action of $\mathcal{B}\Gamma_n$ to be trivial for every element that is the image of a path of $\mathcal{Q}\Gamma_n$ of length ≥ 1 , and let I_v act nontrivially only on the copies of P_v . It is clear that this cannot be simplified further, and has image $\sum_{v \in \text{vert}(\Gamma_n)} J_v(q)[P_v]$ in $K_0(\mathcal{B}\Gamma_n)$. Thus there are no relations among the elements $[P_v]$. \diamond

This provides a strategy for computing the image $[\llbracket T \rrbracket]$, which we will describe presently. First, we spend some times on the idempotent sub-algebra I_{2n} . \diamond

4. THE IDEMPOTENT SUB-ALGEBRA OF $\mathcal{B}\Gamma_n$

The vertices of the quiver defining $\mathcal{B}\Gamma_n$ correspond to certain planar configurations of circles and decorations, called *cleaved links* in [14] and [15]. More specifically, let S be an oriented two-dimensional sphere.

Definition 9. *A cleaved link L in S consists of the following data*

- (1) *a smoothly embedded circle $E_L \subset S$, called the equator for L ,*
- (2) *a marked point $*_L \in E_L$*

- (3) an identification of the closures of the two open discs $S \setminus E_L$ as \overleftarrow{D}_L and \overrightarrow{D}_L , called the inside and outside discs, respectively, where each is oriented from S ,
- (4) the orientation of E_L induced as the boundary of \overleftarrow{D}_L , and
- (5) a (possibly empty) set $\text{CUTCIRCS}(L)$ of $k_L \geq 0$ simple closed curves which each non-trivially and transversely intersect E_L away from $*_L$

Definition 10. For L a cleaved link in an oriented sphere S , P_L is the ordered set whose elements are points of intersection between E_L and $\text{CUTCIRCS}(L)$ equipped with the ordering inherited from the orientation of $E_L \setminus \{*_L\}$. The cardinality of P_L is $2n_L$.

Note that P_L will be ordered opposite the orientation of \overrightarrow{D}_L .

A cleaved link L_1 in a sphere S_1 is equivalent to another cleaved link L_2 in S_2 if there is an orientation preserving diffeomorphism $\phi : S_1 \rightarrow S_2$ which preserves each of the structures in the definition. In particular,

- (1) $\phi(E_{L_1}) = E_{L_2}$ and $\phi(*_{L_1}) = *_{L_2}$,
- (2) $\phi(\overleftarrow{D}_{L_1}) = \overleftarrow{D}_{L_2}$, and thus $\phi(\overrightarrow{D}_{L_1}) = \overrightarrow{D}_{L_2}$
- (3) ϕ maps each $C \in \text{CUTCIRCS}(L_1)$ diffeomorphically to a circle $C' \in \text{CUTCIRCS}(L_2)$

It follows that ϕ induces an order preserving bijection $P_{L_1} \leftrightarrow P_{L_2}$.

Definition 11. A decorated, cleaved link (L, σ) is a cleaved link L and a map $\sigma : \text{CUTCIRCS}(L) \rightarrow \{+, -\}$. The map σ is called the decoration.

Two decorated, cleaved links are equivalent if there is an equivalence ϕ of the undecorated cleaved links with $\phi^*(\sigma_2) = \sigma_1$.

Definition 12. The set of equivalence classes of decorated, cleaved links (L, σ) with $n_L = n$ will be denoted \mathcal{CL}_n .

In $\mathcal{B}\Gamma_n$ there is an idempotent $I_{(L, \sigma)}$ for each equivalence class of decorated, cleaved links. By theorem 8, we know that we should be interested in the $\mathbb{Z}[q^{1/2}, q^{-1/2}]$ modules spanned by these equivalence classes.

Definition 13. For each $n \geq 0$, \mathcal{I}_{2n} is the free $\mathbb{Z}[q^{1/2}, q^{-1/2}]$ -module generated by the elements of \mathcal{CL}_n . The generator corresponding to $(L, \sigma) \in \mathcal{CL}_n$ will be denoted $I_{(L, \sigma)}$.

Examples: When $n = 0$, $\mathcal{I}_0 = \mathbb{Z}[q^{1/2}, q^{-1/2}]$. It is possible to include this in the framework above, by allowing $\text{CUTCIRCS}(L)$ to be empty. Then \mathcal{I}_0 has a generator I_0 corresponding to the equivalence class for $S^2 \subset \mathbb{R}^3$, oriented as the boundary of the unit ball, with $E_L = \{(x, y, 0) | x^2 + y^2 = 1\}$, and $* = (1, 0, 0)$. We take \overleftarrow{D} to be the upper hemisphere, since that endows E_L with the same orientation it inherits from being the boundary of D^2 in $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$. However, we do not obtain a different cleaved link by taking \overleftarrow{D} to be the lower hemisphere since $\phi(x, y, z) = (x, -y, -z)$ is orientation preserving when restricted to S^2 , takes $(1, 0, 0)$ and E to themselves, and carries the upper hemisphere to the lower hemisphere.

Convention for describing generators: Before giving more examples we describe how the choice of $*$, and the ordering of P_L , allows each generator to be identified by combinatorial data. Each generator is determined by two planar matchings of P_L : a planar matching \overleftarrow{m}_L embedded in \overleftarrow{D}_L and the other \overrightarrow{m}_L embedded in \overrightarrow{D}_L . A planar matching on $2n_L$ enumerated points is uniquely determined by a permutation of the even numbers $2, 4, \dots, 2n_L$ by the rule that the k^{th} even number in the permutation $2\sigma_1, \dots, 2\sigma_{n_L}$ is the endpoint $p_{2\sigma_k}$ of the arc starting at p_{2k-1} . We will describe both \overleftarrow{m}_L and \overrightarrow{m}_L by these permutations, as specified by P_L . To finish encoding L we need to specify the decoration \pm on each circle in $\text{CUTCIRCS}(L)$. We do this by first ordering the circles by the order in which we first meet the circles if we start at $*$ and walk around E_L according to its orientation. Thus the circle containing p_1 will always come first in our ordering. This is equivalent to the rule $C < C'$ if, and only if, the smallest subscript of any $p_j \in P_L$ occurring in C is less than the smallest subscript of any p_r occurring in C' . With this ordering, a list of k_L elements from $\{+, -\}$ corresponds to a choice of decorations on the circles of $\text{CUTCIRCS}(L)$: the i^{th} entry in the list is the decoration on the i^{th} circle in the ordering. For example, in Figure 1 the generator B_{-+} is specified by $\overleftarrow{m}_L = (42)$ and $\overrightarrow{m}_L = (42)$. The $-$ decoration occurs on the circle through the point 1, which is the larger circle in the picture, while the smaller, and second circle, is decorated with a $+$.

The generators of \mathcal{I}_2 are cleaved links whose circles intersect its equator exactly twice. For each circle $C \subset S^2$ that intersects another circle E there must be at least two intersections. Consequently $\text{CUTCIRCS}(L) = \{C\}$. C can be decorated with either a $+$ or a $-$. Thus $\mathcal{CL}_2 = \{C_+, C_-\}$ corresponding to this choice of decoration. Then \mathcal{I}_2 has two corresponding generators, which we will write I_+ and I_- , so $\mathcal{I}_2 \cong \mathbb{Z}[q^{1/2}, q^{-1/2}]I_+ \oplus \mathbb{Z}[q^{1/2}, q^{-1/2}]I_-$.

For \mathcal{I}_4 , there are twelve generators, depicted in Figure 1. The inside disc \overleftarrow{D}_L is the shaded disk, while the outside disk \overrightarrow{D}_L is the complement in the sphere. Thus,

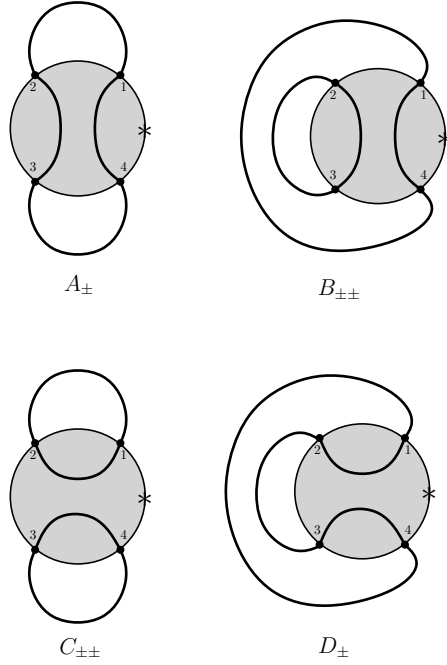


FIGURE 1. The twelve generators of \mathcal{I}_4 grouped based on their inside and outside matchings. There are two generators of type A and D , and four of type B and C , as determined by the choice of decorations.

in terms of the combinatorial data the type D generators have $\overleftarrow{m}_L = (2, 4)$ while $\overrightarrow{m}_L = (4, 2)$.

\mathcal{I}_6 has 104 generators, which will not be listed here.

5. COMPUTING THE DECATEGORYIFICATION OF $\langle\langle T \rangle\rangle$

Let T be an oriented tangle diagram in an oriented two-dimensional disk D_T with a marked point $*_T \in \partial D_T$, and the boundary circle oriented as the boundary of D_T . Let $\text{CR}(T)$ be the set of crossings in T . Let $n_{\pm}(T)$ be the number of positive and negative crossings in T . In this section we explain how to compute $\langle\langle T \rangle\rangle \in K_0(\mathcal{B}\Gamma_n)$ when T has $2n$ endpoints. From the computation in section 3 we know that

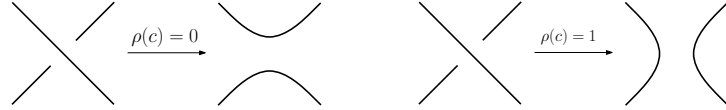
$$\langle\langle T \rangle\rangle = \sum_{(L, \sigma) \in \mathcal{CL}_n} \langle\langle T \rangle\rangle_{I_{(L, \sigma)}} \in \mathcal{I}_{2n}$$

We will describe the generators of $\langle\langle T \rangle\rangle$, and the action of the idempotents, presently. We will also describe the bigradings for each generator. However, there are a few preliminaries before these descriptions.

First, we will consider D_T as being embedded in a sphere S , oriented compatibly

with the orientation of D_T . Let $\vec{D}_T = S \setminus (\text{int } D_T)$. Along with the marked point $*_T$, this effects a decomposition of S into inside and outside discs.

Definition 14. An APS-resolution ρ of T is a map $\rho : \text{CR}(T) \rightarrow \{0, 1\}$. For each APS-resolution, ρ , there is a planar diagram $\rho(T)$ in D^2 , called a APS-resolution diagram, where $\rho(T)$ is the diagram in D^2 obtained by locally replacing (disjoint) neighborhoods of the crossings of T using the following rule for each crossing $c \in \text{CR}(T)$:



Definition 15. A resolution of T is a pair (ρ, \vec{m}) where ρ is an APS-resolution of $T \subset D_T$, and \vec{m} is a planar matching of $P_T = T \cap \partial D_T$ embedded in \vec{D}_T . The resolution diagram $\rho(T, \vec{m})$ for (ρ, \vec{m}) is the diagram in S_T found by gluing $\rho(T)$ to \vec{m} along P_T . The set of resolutions will be denoted $\text{RES}(T)$.

Definition 16. For each $(\rho, \vec{m}) \in \text{RES}(T)$ the homological grading is

$$h(\rho) = \sum_{c \in \text{CR}(T)} \rho(c) - n_-(T)$$

Definition 17. A decorated resolution for T is a resolution (ρ, \vec{m}) and a map $s : \text{CIRCLES}(\rho(T, \vec{m})) \rightarrow \{+, -\}$

The circles in $\rho(T, \vec{m})$ are of two types: 1) *free* circles – those which do not intersect P_T , and 2) *cut* circles – those which do. The set of free circles will be denoted $\text{FREECIRCS}(\rho(T, \vec{m}))$ while the cut circles will be denoted $\text{CUTCIRCS}(\rho(T, \vec{m}))$.

Definition 18. The quantum grading of a decorated resolution (ρ, \vec{m}, s) is

$$I(\rho, \vec{m}, s) = h(\rho) + \sum_{C \in \text{FREECIRCS}(\rho(T, \vec{m}))} s(C) + \frac{1}{2} \sum_{C \in \text{CUTCIRCS}(\rho(T, \vec{m}))} s(C) + (n_+(T) - n_-(T))$$

The bigraded \mathbb{Z} -module obtained from $\llbracket T \rrbracket$ by forgetting the differential, and the module structure over $\mathcal{B}\Gamma_n$, is generated by the decorated resolutions, shifted by the homological and quantum gradings:

$$\llbracket T \rrbracket = \bigoplus_{(\rho, \vec{m}, s)} \mathbb{Z}\{h(\rho), I(\rho, \vec{m}, s)\}$$

To describe the action of an idempotent $I_{(L, \sigma)}$ first notice that there is a map taking a decorated resolution to its *boundary* cleaved circle: $\partial(\rho, \vec{m}, s)$ is the cleaved circle $(\text{CUTCIRCS}(\rho(T, \vec{m})), s|_{\text{cut}})$ where $s|_{\text{cut}}$ is the restriction of s to the cut circles. This map is specified by the choice of marked point $*_T$; different choices will result in different maps.

Then the action of $I_{(L,\sigma)}$ on $\llbracket T \rrbracket$ is the linear extension of

$$(\rho, \vec{m}, s) \cdot I_{(L,\sigma)} = \begin{cases} (\rho, \vec{m}, s) & \partial(\rho, \vec{m}, s) = (L, \sigma) \\ 0 & \text{Otherwise} \end{cases}$$

Consequently, $\llbracket T \rrbracket I_{(L,\sigma)}$ is spanned by the generators whose boundary cleaved circle is exactly (L, σ) .

Since $\llbracket T \rrbracket I_{(L,\sigma)}$ is a bigraded chain complex over \mathbb{Z} , we know how to compute $\llbracket T \rrbracket I_{(L,\Sigma)}$. Let $P_{(L,\Sigma)}$ be the simple module over $\mathcal{B}\Gamma_n$ with \mathbb{Z} in bigrading $(0,0)$ and such that $I_{(L,\sigma)}$ acts by the identity, while the action of the rest of $\mathcal{B}\Gamma_n$ is trivial. Each generator (ρ, \vec{m}, s) gives a copy of $P_{(L,\Sigma)}$ in bigrading $(h(\rho), I(\rho, \vec{m}, s))$. Thus,

$$\llbracket T \rrbracket I_{(L,\Sigma)} = \left(\sum_{\partial(\rho, \vec{m}, s) = (L,\sigma)} (-1)^{h(\rho)} q^{I(\rho, \vec{m}, s)} \right) P_{(L,\sigma)}$$

Under the isomorphism with \mathcal{I}_{2n} , this implies that, in $K_0(\mathcal{B}\Gamma_n)$,

$$\llbracket T \rrbracket = \sum_{(\rho, \vec{m}, s)} (-1)^{h(\rho)} q^{I(\rho, \vec{m}, s)} I_{\partial(\rho, \vec{m}, s)}$$

We illustrate with a few example computations.

Example 1: Suppose T is a tangle in D_T with no boundary, $P_T = \emptyset$. Then $\llbracket T \rrbracket$ is the regular Khovanov homology, and $\llbracket T \rrbracket$ is just $J_T(q)$.

Example 2: Suppose $P_T = \{p_1, p_2\}$ consists of two points, oriented as described in the section on cleaved links. Then the Grothendieck group is \mathcal{I}_2 , which is two dimensional over $\mathbb{Z}[q^{1/2}, q^{-1/2}]$ with basis I_+ and I_- . There is only one right matching which can be used, and resolutions (ρ, \vec{m}, s) come in pairs r_{\pm} depending upon whether the single cut circle is adorned with a $+$ or $-$. Now $h(r_+) = h(r_-)$ and there is a number \tilde{F} such that $I(r_+) = \tilde{F} + 1/2$ while $I(r_-) = \tilde{F} - 1/2$. Consequently, there is a polynomial $F(q)$ for which

$$\llbracket T \rrbracket = F(q)(q^{1/2}I_+ + q^{-1/2}I_-)$$

Example 3: We consider the tangle T_L in Figure 2 which has two arcs and a single positive crossing, with the choice of marked point as illustrated. The 0 resolution consists of two vertical arcs, which can be glued to either outside matching (24) or (42). For $(0, (24), s)$ there is a single circle C and thus two possibilities for s . We then have $h(0, (24), \pm) = 0 - 0 = 0$, since $n_- = 0$, and $I(0, (24), \pm) = 0 + 0 \pm 1/2 + 1 - 0$. Furthermore, $\partial(0, (24), \pm) = A_{\pm}$ in the notation of Figure 1. Thus, these generators contribute terms $q^{3/2}I_{A_+} + q^{1/2}I_{A_-}$. For the matching (42)

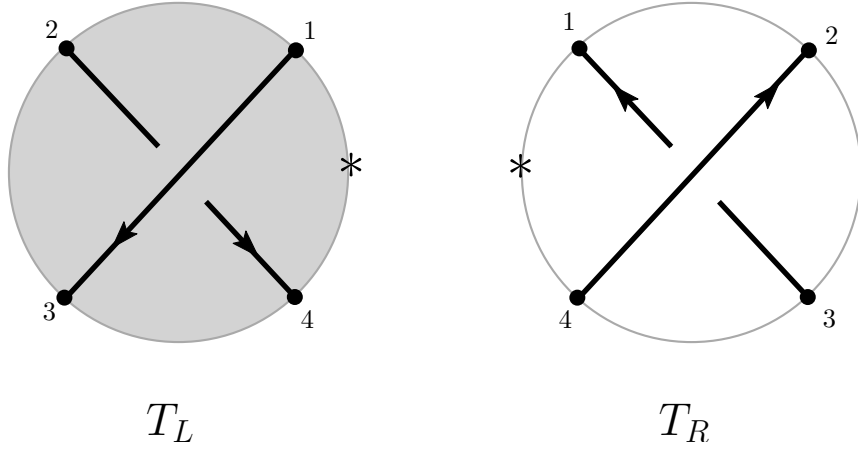


FIGURE 2. The tangle T_L on the left is an inside tangle, since the ordering of its endpoints is with the orientation on the boundary of the disk. It has a single positive crossing. The tangle on the right, T_R , also has a single positive crossing. However, it is an example of an outside tangle.

there are 2 cut circles $B_{\pm\pm}$. Once again, the homological grading is 0, and the quantum grading is $I(0, (42), B_{\pm\pm}) = 0 + 0 \pm 1/2 \pm 1/2 + 1 - 0$, so we have terms $q^2 I_{B_{++}} + q^1(I_{B_{+-}} + I_{B_{-+}}) + I_{B_{--}}$. For the 1 resolution, we have two circles for matching (24) which result in generators $C_{\pm\pm}$. Now the homological grading is 1 and the quantum grading is $I(1, (24), C_{\pm\pm}) = 1 + 0 \pm 1/2 \pm 1/2 + 1 - 0$. Therefore, we obtain terms $-q^3 I_{C_{++}} - q^2(I_{C_{+-}} + I_{C_{-+}}) - q I_{C_{--}}$. Finally, for the (42) matching we have $h(1, (42), D_{\pm}) = 1$ and $I(1, (42), D_{\pm}) = 1 + 0 \pm 1/2 + 1 - 0$, so we have terms $-q^{5/2} I_{D_+} - q^{3/2} I_{D_-}$. In total,

$$\begin{aligned} \llbracket T_L \rrbracket &= q^{3/2} I_{A_+} + q^{1/2} I_{A_-} + q^2 I_{B_{++}} + q^1(I_{B_{+-}} + I_{B_{-+}}) \\ &\quad + I_{B_{--}} - q^3 I_{C_{++}} - q^2(I_{C_{+-}} + I_{C_{-+}}) - q I_{C_{--}} \\ &\quad - q^{5/2} I_{D_+} - q^{3/2} I_{D_-} \end{aligned}$$

Since the homotopy type of $\llbracket T \rrbracket$ is an invariant of the tangle T , the decategorifications in \mathcal{I}_{2n} are also invariants of the tangles. This can be proven directly, and the constructions above generalized, a process which we will return to in the sequel.

6. THE DECATEGORIFICATION OF TYPE D STRUCTURES

Above we mentioned that there are two type of invariants associated to a tangle. More specifically, suppose we have an oriented tangle diagram T in a oriented disc D_T with a marked point $*_T \in \partial D_T$. Let $E = \partial D_T$ be the boundary of D_T with the *opposite* orientation. Let $P_T = \partial D_T \cap T$, ordered according to the orientation on E . Once again we think of D_T as embedded in an oriented sphere S , but now

$\overleftarrow{D}_T = S \setminus (\text{int } D_T)$ is the inside disc for this decomposition of S .

To this configuration is associated a different type of algebraic invariant: a type D structure $\llbracket T \rrbracket$. This is a $(1, 0)$ map $\delta : \llbracket T_2 \rrbracket \longrightarrow \mathcal{B}\Gamma_n \otimes_I \llbracket T \rrbracket$ which satisfies a certain structural identity that will not play a role in this paper. Type D structures also have Grothendieck groups, and, following Ina Petkova in [13] and the results of Lipshitz, Osztvath, and Thurston in [12], the computations are essentially the same as above². Thus, the previous section more or less tells us how to compute the decategorification of the type D structure.

However, the point of introducing the two structures was to obtain a gluing $\llbracket T_1 \rrbracket \boxtimes \llbracket T_2 \rrbracket$, so we will adapt the definition of the decategorification of the type D structure to reflect this pairing in the decategorifications.

We will think of the decategorification of $\llbracket T_1 \rrbracket$ as the map $\mathbb{Z}[q^{1/2}, q^{-1/2}] \longrightarrow K_0(\mathcal{B}\Gamma_n) \cong \mathcal{I}_{2n}$ defined by $f(q) \longrightarrow f(q)\llbracket T_1 \rrbracket$, where $|T_1 \cap \partial D_{T_1}| = 2n$. Accordingly, we will think of the decategorification of $\llbracket T_2 \rrbracket$ as an element in the dual $\mathcal{I}_{2n} \cong K_0(\mathcal{B}\Gamma_n) \longrightarrow \mathbb{Z}[q^{1/2}, q^{-1/2}]$, and thus as an element $\llbracket T_2 \rrbracket$ in $\mathcal{I}_{2n}^* = \text{Hom}_{\mathbb{Z}[q^{\pm 1/2}]}(\mathcal{I}_{2n}, \mathbb{Z}[q^{\pm 1/2}])$.

We will describe this map somewhat tersely: the generators of $\llbracket T_2 \rrbracket$ are the triples $(\rho, \overleftarrow{m}, s)$ where ρ is an APS-resolution of the diagram T_2 , \overleftarrow{m} is a planar matching of P_{T_2} in \overleftarrow{D}_{T_2} (an *inside* matching) and $s : \text{CIRCLES}(\rho(\overleftarrow{m} \# T_2)) \longrightarrow \{+, -\}$. The homological and quantum gradings of this generator are computed identically to those in $\llbracket T_1 \rrbracket$, described above. Likewise, each generator has a boundary $\partial(\rho, \overleftarrow{m}, s) \in \mathcal{CL}_n$ obtained by erasing all the free circles. The action of the idempotent $I_{(L, \Sigma)}$ on the *left* of $\llbracket T_2 \rrbracket$ is trivial on generators whose boundary is different from (L, σ) and the identity on those generators for which (L, σ) is the boundary.

The map $\llbracket T_2 \rrbracket \in \text{Hom}_{\mathbb{Z}[q^{\pm 1/2}]}(\mathcal{I}_{2n}, \mathbb{Z}[q^{\pm 1/2}])$ is the linear extension of the following map on generators $I_{(L, \sigma)}$ of \mathcal{I}_{2n} :

$$\llbracket T_2 \rrbracket(I_{(L, \sigma)}) = \sum_{\partial(\rho, \overleftarrow{m}, s) = (L, \sigma)} (-1)^{h(\rho)} q^{I(\rho, \overleftarrow{m}, s)}$$

Example: We consider the tangle T_R in Figure 2 which is an outside disc containing a single positive crossing. To specify the map $\llbracket T_R \rrbracket$ we need to specify the image on each of the generators in Figure 1. Note, however, that the arcs coming from resolving the crossing will be those *outside* the shaded discs in Figure 1. Thus a 0-resolution of the crossing gives (42) as the *outside* matching. If we choose (24) for the inside matching, then we obtain the generators D_{\pm} . For D_+ there is one positive

²The various bimodule constructions of [12] showing the equivalence of categories can be adapted to the bordered Khovanov setting

circle, so $I(D_+) = 0 + 1/2 + 1$ and $h(D_+) = 0 - 0$. Thus $[[T_R]](D_+) = q^{1/2+1} = q^{3/2}$. However, if we choose (42) as the inside matching, then we obtain the generators $B_{\pm\pm}$. Since B_{-+} has $h(B_{-+}) = 0 - 0$ and $I(B_{-+}) = 0 - 1/2 + 1/2 + 1 = 1$ we can compute $[[T_R]](B_{-+}) = q$. To obtain C_{++} we need to choose (24) as both the inside and outside matching. Thus we need the 1 resolution of the tangle diagram. Then $h(C_{++}) = 1$ and $I(C_{++}) = 1 + 1/2 + 1/2 + 1 = 3$. Similar computations for each of the generators produce the following map:

$$\begin{array}{lll} A_+ \longrightarrow -q^{5/2} & B_{-+} \longrightarrow q & C_{-+} \longrightarrow -q^2 \\ A_- \longrightarrow -q^{3/2} & B_{--} \longrightarrow 1 & C_{--} \longrightarrow -q \\ B_{++} \longrightarrow q^2 & C_{++} \longrightarrow -q^3 & D_+ \longrightarrow q^{3/2} \\ B_{+-} \longrightarrow q & C_{+-} \longrightarrow -q^2 & D_- \longrightarrow q^{1/2} \end{array}$$

Note that the map $[[T]]$ is in principal determined by the image of the cleaved links with all $+$ decorations, since changing a $+$ to a $-$ does not change the APS-resolutions and matching for those states. It does multiply by q^{-1} due to the change in quantum grading. Thus, once we know that $B_{++} \longrightarrow q^2$ we know that $B_{-+} \longrightarrow q^{-1}(q^2)$, for example.

7. RECOVERING THE JONES POLYNOMIAL

Suppose S is an oriented sphere decomposed as $\overleftarrow{D} \cup \overrightarrow{D}$, where \overleftarrow{D} and \overrightarrow{D} are two oriented discs who intersect only on their common boundary, and that boundary is oriented as the boundary of \overleftarrow{D} . Let $* \in \partial \overleftarrow{D}$. Suppose further that L is an oriented link diagram in S which intersects $\partial \overleftarrow{D}$ transversely away from its crossings. Then $T_1 = L \cap \overleftarrow{D}$ and $T_2 = L \cap \overrightarrow{D}$ are an inside tangle and an outside tangle, respectively.

Proposition 19. *The composition*

$$\mathbb{Z}[q^{1/2}, q^{-1/2}] \xrightarrow{[[T_1]]} \mathcal{I}_{2n} \xrightarrow{[[T_2]]} \mathbb{Z}[q^{1/2}, q^{-1/2}]$$

equals multiplication by the Jones polynomial $J_L(q)$, described in the introduction.

Proof: $J_L(q)$ is the sum $\sum_{(\rho,s)} (-1)^{h(\rho)} q^{I_J(\rho,s)}$ where 1) $\rho : \text{CR}(L) \longrightarrow \{0, 1\}$ is a resolution of the diagram L with diagram $\rho(L)$ found using same rules as for APS-resolutions, and 2) $s : \text{CIRCLES}(\rho(L)) \longrightarrow \{+, -\}$ as above. The value $h(\rho)$ is computed as above, using all the crossings in L . $I_J(\rho, s)$ is computed as above, noting that there will be no cut circles in $\rho(L)$ so every circle will contribute ± 1 to the count.

For each such resolution diagram $\rho(L)$, let $\overleftarrow{\rho}(L) = \rho(L) \cap \overleftarrow{D}$. Then $\overleftarrow{\rho}(L)$ is an APS-resolution diagram for T_1 , whose resolution is simply the restriction of ρ to the crossings in \overleftarrow{D} . A similar story holds for $\rho(L) \cap \overrightarrow{D} = \overrightarrow{\rho}(L)$. Let \overrightarrow{m} be the planar matching obtained by deleting the free circles from $\overrightarrow{\rho}(L)$ and let \overleftarrow{s} be the decoration on $\overleftarrow{\rho} \# \overrightarrow{m}$ found by restricting s . Then $(\overleftarrow{\rho}, \overrightarrow{m}, \overleftarrow{s})$ is a generator for $[[T_1]]$. If

we take \overleftarrow{m} to be the planar matching found by deleting the free circles of $\overleftarrow{\rho}(L)$ then $(\overrightarrow{\rho}, \overleftarrow{m}, \overrightarrow{s})$ is a generator of $\llbracket T_2 \rrbracket$.

Thus every generator ξ of $\langle\langle L \rangle\rangle$ maps to a pair $(\overleftarrow{\xi}, \overrightarrow{\xi})$ of generators from $\llbracket T_1 \rrbracket$ and $\llbracket T_2 \rrbracket$. Furthermore, $\partial \overleftarrow{\xi} = \partial \overrightarrow{\xi} = (\overleftarrow{m} \# \overrightarrow{m}, \sigma)$ where σ is the restriction of s to the circles of $\rho(L)$ intersecting $\partial \overleftarrow{D}$. This is illustrated in Figure 3. In fact, given any such pair of generators $(\overleftarrow{\xi}, \overrightarrow{\xi})$ with $\partial \overleftarrow{\xi} = \partial \overrightarrow{\xi}$ there is a unique generator ξ of $\langle\langle L \rangle\rangle$ which maps to the pair. Thus, this construction generates a bijection between the generators of $\langle\langle L \rangle\rangle$ and the set of pairs $(\overleftarrow{\xi}, \overrightarrow{\xi})$ with $\partial \overleftarrow{\xi} = \partial \overrightarrow{\xi}$.

It is straightforward to check that, under this bijection, $h(\rho) = h(\overleftarrow{\rho}) + h(\overrightarrow{\rho})$, since the crossings of L are partitioned between \overleftarrow{D} and \overrightarrow{D} . It is also true that $I_J(\rho, s) = I(\overleftarrow{\rho}, \overrightarrow{m}, \overleftarrow{s}) + I(\overrightarrow{\rho}, \overleftarrow{m}, \overrightarrow{s})$ since a \pm cut circle will be counted as $\pm 1/2$ in the sum defining each of $I(\overleftarrow{\rho}, \overrightarrow{m}, \overleftarrow{s})$ and $I(\overrightarrow{\rho}, \overleftarrow{m}, \overrightarrow{s})$, but as ± 1 in the sum for $I_J(\rho, s)$. Thus the monomial in $J_L(q)$ for each generator ξ of $\langle\langle L \rangle\rangle$ is a product

$$(-1)^{h(\xi)} q^{I_J(\xi)} = (-1)^{h(\overleftarrow{\xi})} q^{I(\overleftarrow{\xi})} (-1)^{h(\overrightarrow{\xi})} q^{I(\overrightarrow{\xi})}$$

where the factors on the right are monomial terms in $\llbracket T_1 \rrbracket$ and $\llbracket T_2 \rrbracket$. Furthermore, provided that $\partial \overleftarrow{\xi} = \partial \overrightarrow{\xi}$ the product of the terms on the right will equal a monomial in $J_L(q)$.

Let $I_{(L,\sigma)}^*$ be the Kronecker dual functional for $I_{(L,\sigma)}$. Then the composition

$$\mathbb{Z}[q^{1/2}, q^{-1/2}] \xrightarrow{\llbracket T_1 \rrbracket} \mathcal{I}_{2n} \xrightarrow{\llbracket T_2 \rrbracket} \mathbb{Z}[q^{1/2}, q^{-1/2}]$$

is multiplication by the polynomial that results from

$$\begin{aligned} & \left(\sum_{I_{(L,\sigma)}} \left(\sum_{\partial(\overrightarrow{\rho}, \overleftarrow{m}, s') = (L,\sigma)} (-1)^{h(\overrightarrow{\rho})} q^{I(\overrightarrow{\rho}, \overleftarrow{m}, s')} \right) I_{(L,\sigma)}^* \right) \circ \left(\sum_{I_{(L,\sigma)}} \left(\sum_{\partial(\overleftarrow{\rho}, \overrightarrow{m}, s) = (L,\sigma)} (-1)^{h(\overleftarrow{\rho})} q^{I(\overleftarrow{\rho}, \overrightarrow{m}, s)} \right) I_{(L,\sigma)} \right) \\ &= \sum_{\partial \overleftarrow{\xi} = \partial \overrightarrow{\xi}} (-1)^{h(\overleftarrow{\xi})} q^{I(\overleftarrow{\xi})} (-1)^{h(\overrightarrow{\xi})} q^{I(\overrightarrow{\xi})} \end{aligned}$$

where $\overleftarrow{\xi}$ and $\overrightarrow{\xi}$ are allowed to range over all generators of $\llbracket T_1 \rrbracket$ and $\llbracket T_2 \rrbracket$.

From our argument above, the terms in this sum are in one-to-one correspondence with the terms in the sum defining $J_L(q)$, and equal the corresponding term. Thus, the sum is equal to $J_L(q)$. \diamond

Example: We can glue the discs containing T_L and T_R along their boundaries so

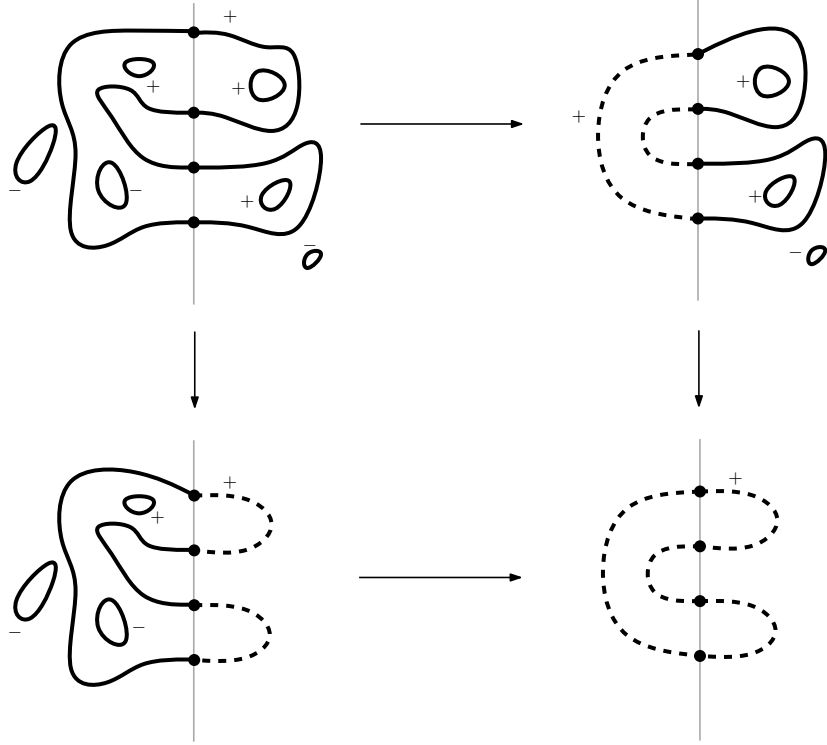


FIGURE 3. The top left diagram represents a decorated resolution ξ used to compute $J_L(q)$. The bottom left is the corresponding generator in $\langle\langle T_1 \rangle\rangle$. The dashed arcs in the bottom left represent the outside matching. They are the arcs which remain when we delete all the free circles to the right of the vertical line. The top right is the generator used in $\langle\langle T_2 \rangle\rangle$. The bottom right is the cleaved link that is the common boundary of these generators. Notice that the bottom right diagram is determined solely by the cut circles, and their decorations, shown in the top left diagram. Notice also that the top right and bottom left can be identified along their boundary to recover the diagram in the top left, simply by laying one diagram atop the other so that the boundaries match.

that the marked and labeled points are identified. This gives a link diagram for the

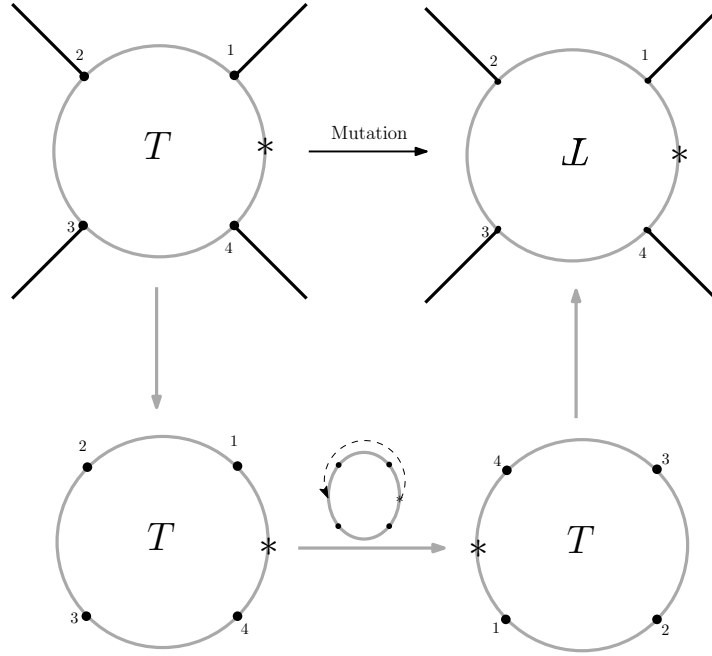


FIGURE 4. The top row is a mutation of the diagram, where we have only depicted the region around the tangle T . We have chosen a marked point and labeled the intersections with the boundary of the local disc, so that the local disc is an inside disc. The bottom left is the inside disc. Moving the marked point two segments counter-clockwise is shown in the lower row. This relabels the intersection points as well. Gluing the new inside disc according to the boundary data produces the mutant.

positive Hopf link. Computing $\llbracket T_R \rrbracket \circ \llbracket T_L \rrbracket$ gives

$$\begin{aligned}
 & \llbracket T_R \rrbracket (q^{3/2}I_{A+} + q^{1/2}I_{A-} + q^2I_{B++} + q(I_{B+-} + I_{B-+}) + I_{B--} \\
 & \quad - q^3I_{C++} - q^2(I_{C+-} - I_{C-+}) - qI_{C--} - q^{5/2}I_{D+} - q^{3/2}I_{D-}) \\
 & = q^{3/2}(-q^{5/2}) + q^{1/2}(-q^{3/2}) + q^2(q^2) + q(q + q) \\
 & \quad + 1 - q^3(-q^3) - q^2(-q^2 - q^2) - q(-q) - q^{5/2}(q^{3/2}) - q^{3/2}(q^{1/2}) \\
 & = -q^4 - q^2 + q^4 + 2q^2 + 1 + q^6 + 2q^4 + q^2 - q^4 - q^2 \\
 & = q^6 + q^4 + q^2 + 1
 \end{aligned}$$

This is the correct polynomial $J_L(q)$ for this Hopf link.

8. MUTATION INVARIANCE

We use the results of the last section to provide an alternative proof of the well known fact that $J_L(q)$ is invariant under mutations of L . A diagram L' is a mutant of a diagram L if there is a disc D in L which intersects L in 4 points, such that L' is obtained by removing D and gluing it back with a 180° twist about one of the three principal axes in \mathbb{R}^3 . Using the constructions in this paper, we can describe mutation in terms of moving the marked point $*$ used in gluing the local disc. Pick $*$ on ∂D and let $T_1 = L \cap D$ and $T_2 = L \cap (S \setminus D)$. With the marked point, T_1 is an inside tangle in D , and T_2 is an outside tangle. To obtain L' we move $*$ on ∂D two segments counter-clockwise, while keeping the marked point on $\partial(S \setminus D)$ fixed. This process, and how it achieves mutation are shown in Figure 4. Gluing to align the marked points results in the mutant diagram L' .

In [17], it is shown that this one type of mutation is enough to obtain mutations where we rotate the diagram in D around the other two perpendicular axes. These rotations do not have easy interpretations in terms of the marked point.

Consider the element $[[T_1]] \in \mathcal{I}_4$. Given a generator (ρ, \vec{m}, s) of $\langle\langle T_1 \rangle\rangle$, moving the marked point $*$ changes $\partial(\rho, \vec{m}, s)$ but not $h(\rho)$, hence we will reverse orientations on all the strands and thus not change any of the crossings, or $I(\rho, \vec{m}, s)$. Consequently, we can think of moving $*$ as providing a map $M_* : \mathcal{I}_4 \rightarrow \mathcal{I}_4$ under which $[[T_*]]$ is $M_*[[T]]$ if T_* is the inside tangle with the new marked point. From the generators in Figure 1 we can compute the effect on the generators of \mathcal{I}_4 of moving $*$ *one* segment counter-clockwise:

$$\begin{array}{llll} A_\pm \longrightarrow D_\pm & D_\pm \longrightarrow A_\pm & B_{++} \longrightarrow C_{++} & C_{++} \longrightarrow B_{++} \\ B_{--} \longrightarrow C_{--} & C_{--} \longrightarrow B_{--} & C_{+-} \longrightarrow B_{+-} & C_{-+} \longrightarrow B_{-+} \\ B_{+-} \longrightarrow C_{-+} & B_{-+} \longrightarrow C_{+-} & & \end{array}$$

Notice that the signs in $B_{+-} \rightarrow C_{-+}$ are switched. Recall that the circles in B_{+-} are ordered by the lowest number appearing on the circle. This number can change when we move $*$, but the decoration on the circle does not.

To compute M_* we must move $*$ two segments. This produces a map on \mathcal{I}_4 which fixes all the generators except $B_{+-} \leftrightarrow B_{-+}$ and $C_{+-} \leftrightarrow C_{-+}$. Thus, to prove mutation invariance it suffices to see that the coefficient of B_{+-} in $[[T_1]]$ is the same as B_{-+} , and likewise for C_{+-} and C_{-+} . For then, when we apply $[[T_2]]$ to $\langle\langle T_1 \rangle\rangle$ we will get the same result as applying it to $M_*^2[[T_1]]$. As these compositions compute the Jones polynomial for L and its mutant L' , we will have verified the mutation invariance.

Suppose we have a generator (ρ, \vec{m}, s) of $\langle\langle T_1 \rangle\rangle$ with $\partial(\rho, \vec{m}, s) = B_{-+}$. Then we

can find a corresponding generator (ρ', \vec{m}', s') whose boundary is B_{+-} . We take $\rho' = \rho$ and $\vec{m}' = \vec{m}$, and thus have the same resolution diagram. All that changes is that s' will evaluate on the cut circles opposite how s does. Therefore, $h(\rho) = h(\rho')$ and $I(\rho, \vec{m}, s) = I + 1/2 - 1/2$, where we have pulled out the contribution of the cut circles, while $I(\rho', \vec{m}', s') = I - 1/2 + 1/2$. As these are equal, the coefficient on B_{-+} from (ρ, \vec{m}, s) is identical to that on B_{+-} from (ρ', \vec{m}', s') . Since the map $(\rho, \vec{m}, s) \rightarrow (\rho', \vec{m}', s')$ is evidently a bijection from the generators with boundary B_{-+} to those with boundary B_{+-} , we can conclude that the coefficient on B_{-+} in $[[T_1]]$ is identical to that on B_{+-} . A similar argument shows that the coefficients on C_{+-} and C_{-+} are also equal. \diamond

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